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SOLVING SECOND KIND SINGULAR INTEGRAL EQUATIONS USING CHEBYSHEV POLYNOMIALS

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Abstract: *A numerical developed technique to solve Fredholm integral equation of the second kind with separable singular kernel is proposed. This technique relies on the truncated expansion functions of the kernels in the finite series of the weighted Chebyshev polynomials of first, second, third, and fourth kinds. Three numerical examples are presented for verification and validation of the developed technique. The results showed that even with small n, the numerical results are accurate.*
Keywords*: Singular integral equations, singular kernel, Cauchy singularity, Chebyshev polynomial, weight function accuracy.*

INTRODUCTION

$$\alpha(x)u(x) + \frac{\beta(x)}{\pi}\left(⨍_{-1}^{1}\frac{k(t,x)u(t)}{t-x}dt + \int_{-1}^{1}Q(t,x)u(t)dt\right) = f(x),\ -1 < x < 1 \quad (1)$$

where $k(t,x)$ and Q(t, x) are real v alued functions which

Generally, Cauchy singular integral equations of the second kind can be expressed in the form: singularity of the Cauchy-type. The Cauchy singular integral equations are encountered in a variety of mixed boundary value problems in mathematical physics such as fracture problems in solid mechanics (Ladopoulos, 2000), aerodynamics and plane elasticity (Kalandiya, 1975) and other related problems. Several numerical techniques have been used to solve Cauchy singular integral equations including polynomials like the weighted Chebyshev polynomial of the second satisfy the Hölder condition with respect to each of the kind (Eshkuvatov et al., 2012), Bernstein polynomial independent variables, and
$f(x), \alpha(x)$ $\beta(x)$
are square method (Setia, 2014), using Legendre polynomial (Setia integrable functions on the interval
$[-1,1]$, $\alpha(x) \neq 0, \beta(x) \neq 0$ $x \in [-1,1]$ $u(x)$
et al., 2015), the reproducing kernel Hilbert space method for any and (Dezhbord et al., 2016), and the collocation technique
is the solution to be determined. Equation 1 has based on the Bernstein polynomials (Seifi et al., 2017).

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In this research work, we present a developed technique for solution of Cauchy singular integral equation by using the weighted Chebyshev polynomials of the first, second, third and fourth kinds. The used approximated method for solving Equation 1 stems from the work of Eshkuvatov et al. (2012) wherein approximate method has been developed to solve the case for k(t,x)=1 and Q(x,t)=0 using the weighted Chebyshev polynomial of the second kind only.

THE CHEBYSHEV POLYNOMIAL TECHNIQUE

To be any of the four Chebyshev implies that is the set of weight functions of the Chebyshev polynomials of the first, second, third and fourth kinds, respectively and we denote these weight functions by

$\{w^{(1)}(x), w^{(2)}(x), w^{(3)}(x), w^{(4)}(x)\}$

$$u(x) = w(x)\,h(x), \tag{2}$$

where $h(x)$ is some bounded function of x on the interval $[-1,1]$ and $w(x) \in \{w^{(1)}(x), w^{(2)}(x), w^{(3)}(x), w^{(4)}(x)\}$ is the given weight function. Approximating $h(x)$ in Equation 2 by using the Chebyshev polynomial gives:

The integrals in Equation 5 can be calc ulated given that the kernel $k(t,x)$ and $Q(t,x)$ can be expressed in the form (Dardery and Allan, 2013):

$$k(t,x) = \sum_{i=0}^{m} k_i(x)t^i, \qquad Q(t,x) = \sum_{q=0}^{s} Q_q(x)t^q \tag{6}$$

with the express ions $k_i(x)$ and $Q_q(x)$ known. The t^i and t^q are expressed in terms of the Chebyshev polynomials of the first kind, $T_i(t)$, of degree up to i (Mason and Handscomb, 2003):

$$t^i = 2^{1-i}\sum_{n=0}^{\left[\frac{i}{2}\right]}{}' \, {}^{i}c_n\, T_{i-2n}(t), \tag{7}$$

where the dash $(\sum')$ denotes that the $\frac{i}{2}^{th}$ term in the sum is to be halved if i is even and $n = \frac{i}{2}$. By making use of Equation 6, we can represent Equation 5 as:

$$\sum_{k=0}^{n} a_k\left(w(x)\alpha(x)H_k(x) + \frac{\beta(x)}{\pi}\sum_{i=0}^{m} k_i(x)\int_{-1}^{1}\frac{w(t)\,t^iH_k(t)}{t-x}dt\right) + \sum_{k=0}^{n} a_k\frac{\beta(x)}{\pi}\sum_{q=0}^{s} Q_q(x)\int_{-1}^{1} w(t)t^qH_k(t)dt = f(x) \tag{8}$$

Define

(9)

We have

(10)

With the help of the properties of the Chebyshev polynomials (Mason and Handscomb, 2003), we have:

$$\left.\begin{aligned}
T_i(t)U_k(t) &= \frac{1}{2}\left(U_{k+i}(t) + U_{k-i}(t)\right), && k \geq i-1\\
T_i(t)U_k(t) &= \frac{1}{2}\left(U_{k+i}(t) - U_{i-k}(t)\right), && k < i-1\\
T_i(t)V_k(t) &= \frac{1}{2}\left(V_{k+i}(t) + V_{k-i}(t)\right), && k \geq i\\
T_i(t)V_k(t) &= \frac{1}{2}\left(V_{k+i}(t) + V_{i-k-1}(t)\right), && k < i\\
T_i(t)W_k(t) &= \frac{1}{2}\left(W_{k+i}(t) + W_{k-i}(t)\right), && k \geq i\\
T_i(t)W_k(t) &= \frac{1}{2}\left(W_{k+i}(t) + W_{i-k-1}(t)\right), && k < i\\
T_i(t)T_k(t) &= \frac{1}{2}\left(T_{k+i}(t) + T_{|k-i|}(t)\right) &&
\end{aligned}\right\} \tag{11}$$

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$$h(x) \approx h_n(x) = \sum_{k=0}^{n} a_k H_k(x) \quad (3)$$

where we can represent the unknown function as:

$$u(x) \approx w(x) \sum_{k=0}^{n} a_k H_k(x), \quad -1 < x < 1, \quad (4)$$

where a_k are unknown coefficients to be determined. Substituting the approximate solution (Equation 4) for the

$$\lambda_k(x) = \sum_{i=0}^{m} k_i(x) ⨍_{-1}^{1} \frac{w(t)\,t^i H_k(t)}{t-x} dt + \sum_{q=0}^{s} Q_q(x) \int_{-1}^{1} w(t)\,t^q H_k(t)\,dt,$$

$$\sum_{k=0}^{n} a_k \left(w(x)\,\alpha(x)\,H_k(x) + \lambda_k(x) \frac{\beta(x)}{\pi} \right) = f(x)$$

We can also recall the following relations (Dardery and Allan, 2013):

$$\int_{-1}^{1} \frac{T_k(x)\,T_j(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & k \neq j \\ \pi, & k = j = 0 \\ \frac{\pi}{2}, & k = j \neq 0 \end{cases} \quad (12)$$

$$\int_{-1}^{1} \sqrt{1-x^2}\,U_k(x)\,U_j(x)\,dx = \begin{cases} 0, & k \neq j \\ \frac{\pi}{2}, & k = j \end{cases} \quad (13)$$

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,V_k(x) V_j(x)\,dx = \begin{cases} 0, & k \neq j \\ \pi, & k = j \end{cases} \quad (14)$$

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\,W_k(x) W_j(x)\,dx = \begin{cases} 0, & k \neq j \\ \pi, & k = j \end{cases} \quad (15)$$

Using the expressions (Equation 11) in Equation 9, and the defined relations (Equations 12, 13, 14 and 15), we can obtain an exact value for λ_k. By multiplying each of the terms in Equation 10 by the Chebyshev polynomials of the first kind, $T_j(x), j = 0,1,\cdots,n$ and integrate from -1 to 1, we obtain the equation:

$$\sum_{k=0}^{n} a_k \left(\int_{-1}^{1} w(x)\,\alpha(x) H_k(x) T_j(x)\,dx + \frac{1}{\pi} \int_{-1}^{1} \lambda_k(x) \beta(x) T_j(x)\,dx \right) = \int_{-1}^{1} f(x) T_j(x)\,dx \quad (16)$$

(Equation 20), the constants B_{kj} Equation 19 have the following forms:

$$\left.\begin{aligned} B_{kj} &= \sum_{l=0}^{m}{}' b_l \int_{-1}^{1} w(x) H_k(x) T_j(x) T_l(x)\, dx \\ C_{kj} &= \sum_{l=0}^{m}{}' c_l \int_{-1}^{1} \lambda(x) T_j(x) T_l(x)\, dx \end{aligned}\right\}$$

written in the form:

$$\sum_{k=0}^{n} a_k A_{kj} = d_j, \quad j = 0,1,\cdots,n, \qquad (18)$$

Lemma 1

For $k,\ j,\ l\ =\ 0,1,2,\ \ldots$

$$\int_{-1}^{1} w(x) H_k(x) T_j(x) T_l(x)\, dx = \frac{1}{2}\left(\int_{-1}^{1} w(x) H_k(x) T_{j+l}(x)\, dx + \int_{-1}^{1} w(x) H_k(x) T_{|j-l|}(x)\, dx\right) \qquad (23)$$

By applying the relations defined in Equations 12, 13, 14 and 15 and the known fact (Kythe and Schäferkotter, 2004):

$$\int_{-1}^{1} T_n(x)\, dx = \begin{cases} \dfrac{2}{1-n^2}, & n \ \text{even} \\ 0, & n \ \text{odd} \end{cases} \qquad (17)$$

$$\left.\begin{aligned} A_{kj} &= B_{kj} + C_{kj} \\ B_{kj} &= \int_{-1}^{1} w(x)\, \alpha(x) H_k(x) T_j(x)\, dx \\ C_{kj} &= \frac{1}{\pi}\int_{-1}^{1} \lambda_k(x) \beta(x) T_j(x)\, dx \\ d_j &= \int_{-1}^{1} f(x) T_j(x)\, dx \end{aligned}\right\}$$

we obtain a system of linear equations which can be Nfor and Ndukum 29

The integrals in B_{kj} and C_{kj} (19)

defined in Equation 19 can

be calculated exactly by expanding the functions $\alpha(x)$ in the Chebyshev truncated series of the first

and $\beta(x)$ kind:

(20)

$$\left.\begin{aligned} \alpha(x) &= \sum_{l=0}^{m}{}' b_l T_l(x) \\ \beta(x) &= \sum_{l=0}^{m}{}' c_l T_l(x) \end{aligned}\right\}$$

where

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$$\left.\begin{aligned} b_l &= \frac{2}{\pi}\int_{-1}^{1}\frac{\alpha(x)}{\sqrt{1-x^2}}T_l(x)dx \\ c_l &= \frac{2}{\pi}\int_{-1}^{1}\frac{\beta(x)}{\sqrt{1-x^2}}T_l(x)dx \end{aligned}\right\} \qquad (21)$$

and the prime denotes that half of the first term in the sum has been considered. By making use of system and C_{kj} defined in

(22)

$$\int_{-1}^{1} T_k(x)T_j(x)T_l(x)dx = \frac{1}{4}\left(I_1(k,j,l) + I_2(k,j,l) + I_3(k,j,l) + I_4(k,j,l)\right), \qquad (24)$$

where with reference to Equations 11 and 17, we have:

$$I_1(k,j,l) = \begin{cases} \dfrac{2}{1-(k+j+l)^2}, & k+j+l \text{ is even} \\ 0, & k+j+l \quad \text{odd} \end{cases}$$

$$I_2(k,j,l) = \begin{cases} \dfrac{2}{1-(k-j-l)^2}, & |k-j-l| \text{ is even} \\ 0, & |k-j-l| \quad \text{odd} \end{cases}$$

$$I_3(k,j,l) = \begin{cases} \dfrac{2}{1-(k+|j-l|)^2}, & k+|j-l| \text{ is even} \\ 0, & k+|j-l| \quad \text{odd} \end{cases}$$

$$I_4(k,j,l) = \begin{cases} \dfrac{2}{1-(k-|j-l|)^2}, & |k-|j-l|| \text{ is even} \\ 0, & k-|j-l|| \quad \text{odd} \end{cases}$$

Proof

By making use of the last equation in relation (Equation 11) on the left-hand side of Equation 24, we get:

and the two integrals on the right-hand side of Equation 11) on the left-hand side of Equation 23, we get the 23 can be calculated exactly by making use of Equation expression on the right-hand side of Equation 23. 11 and the relations (Equations 12, 13, 14 and 15).

Proof **Lemma 2**

By making use of the last equation in relation (Equation For $k,\ j,\ l\ =\ 0, 1, 2, \ldots$

$$\int_{-1}^{1} T_k(x)T_j(x)T_l(x)dx = \frac{1}{2}\int_{-1}^{1} T_k(x)\left(T_{j+l}(x) + T_{|j-l|}(x)\right)dx = \frac{1}{2}\left(\int_{-1}^{1} T_k(x)T_{j+l}(x)dx + \int_{-1}^{1} T_k(x)T_{|j-l|}(x)dx\right)$$
$$= \frac{1}{4}\left(\int_{-1}^{1}(T_{k+j+l}(x) + T_{|k-j-l|}(x))dx + \int_{-1}^{1}(T_{k+|j-l|}(x) + T_{|k-|j-l||}(x))dx\right)$$

We can now apply the relation defined in Equation 17 to the last expression in the earlier stater equation and desired result is obtained.

By using Lemma 1 and Lemma 2 in Equation 22 and then substitute Equation 22 into equation 18, we obtain a system of linear equations to solve for the unknown coefficients $a_k,\ k = 0,1,\ldots,n$. By substituting the values of a_k into Equation 4, we obtain the numerical solution of the Equation 1.

NUMERICAL EXAMPLES

Here, we apply the numerical technique explained in the previously.

Example 1 Consider the following singular integral equation:

$$u(x) - \frac{1}{\pi}\fint_{-1}^{1} \frac{xtu(t)}{t-x}\, dt = f(x)\,, \quad (25)$$

where $f(x) = \sqrt{1-x^2}U_2(x) + 4x^5 - 3x^2$.

Solution

The analytical solution to Equation 25 is:

$$u(x) = \sqrt{1-x^2}(4x^2-1)\,. \quad (26)$$

Solving the integral Equation 25 using our developed technique, we set $n = 3$ and the unknown function as:

$$u(x) = \sqrt{1-x^2}\sum_{k=0}^{3} a_k U_k(x),$$

where $U_K(x)$ is the Chebyshev polynomial of the second kind. The Chebyshev polynomial of the second kind is chosen since the function f(x) in Equation 25 has the weight function of Chebyshev polynomial of the second kind. By substituting Equation 27 into Equation 25, we get:

$$\sum_{k=0}^{3} a_k\left(\sqrt{1-x^2}U_k(x) - \frac{x}{\pi}\fint_{-1}^{1}\frac{t\sqrt{1-t^2}U_k(t)}{t-x}dt\right) = f(x), \quad (28)$$

From Equations 7 and 11, it follows that:

By applying the properties of Chebyshev polynomials in Equation 29 and simplify, we obtain:

$$\sum_{k=0}^{3} a_k\sqrt{1-x^2}U_k(x) - \frac{x}{2\pi}\left(a_0\fint_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}U_1(t)dt + a_1\fint_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}(U_2(t)+U_0(t))dt\right)$$
$$-\frac{x}{2\pi}\left(a_2\fint_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}(U_3(t)+U_1(t))dt + a_3\fint_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}(U_4(t)+U_2(t))dt\right) = f(x) \quad (29)$$

Substituting the values in Equation 33 into Equation 32 gives a linear system of equations which can be written in matrix form as:

$$\begin{pmatrix} \frac{\pi}{2} & \frac{2}{15} & 0 & -\frac{26}{105} \\ \frac{1}{15} & \frac{\pi}{4}+\frac{1}{15} & -\frac{2}{35} & 0 \\ -\frac{\pi}{4} & \frac{22}{105} & \frac{\pi}{4} & \frac{2}{63} \\ \frac{1}{7} & 0 & \frac{94}{315} & \frac{\pi}{4} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{35} \\ \frac{\pi}{4} \\ \frac{94}{315} \end{pmatrix} \quad (34)$$

Solving the Equation 34, using simple Matlab command, we obtained:

$$a_0 = -0.000000000000000$$
$$a_1 = 0$$
$$a_2 = 1.000000000000000$$
$$a_3 = -0.000000000000000 \quad (35)$$

Substituting the values of $a_k, k = 0,1,2,3$ into Equation 27, the numerical solution of Equation 25 is obtained to be defined in Okecha and Onwukwe (2012) on the integrals

$$\sum_{k=0}^{3} a_k\sqrt{1-x^2}U_k(x) + \frac{x}{2}\left(a_1T_1(x) + (a_0+a_2)T_2(x) + (a_1+a_3)T_3(x) + a_2T_4(x) + a_3T_5(x)\right) = f(x), \quad (30)$$

Multiply Equation 30 through by $T_j(x),\ j = 0,1,2,3,$ integrate from -1 to 1 and make use of relation 7, we have:

$$\begin{aligned}\sum_{k=0}^{3} a_k\int_{-1}^{1}\sqrt{1-x^2}U_k(x)T_j(x)dx &+ \frac{a_1}{2}\int_{-1}^{1}T_1(x)T_1(x)T_j(x)\,dx + \frac{(a_0+a_2)}{2}\int_{-1}^{1}T_1(x)T_2(x)T_j(x)dx \\ &+ \frac{(a_1+a_3)}{2}\int_{-1}^{1}T_1(x)T_3(x)T_j(x)dx + \frac{a_2}{2}\int_{-1}^{1}T_1(x)T_4(x)T_j(x)dx + \frac{a_3}{2}\int_{-1}^{1}T_1(x)T_5(x)T_j(x)dx \\ &= \int_{-1}^{1}f(x)T_j(x)dx,\end{aligned} \quad (31)$$

From Equation 31, we obtain the system of linear

where

$$A_{kj} = a_k\int_{-1}^{1}\sqrt{1-x^2}U_k(x)T_j(x)dx$$

$$B_j = \frac{a_1}{2}\int_{-1}^{1}T_1(x)T_1(x)T_j(x)dx$$

$$C_j = \frac{(a_0+a_2)}{2}\int_{-1}^{1}T_1(x)T_2(x)T_j(x)dx$$

$$D_j = \frac{(a_1+a_3)}{2}\int_{-1}^{1}T_1(x)T_3(x)T_j(x)dx$$

$$E_j = \frac{a_2}{2}\int_{-1}^{1}T_1(x)T_4(x)T_j(x)dx$$

$$F_j = \frac{a_3}{2}\int_{-1}^{1}T_1(x)T_5(x)T_j(x)\,dx,$$

$$G_j = \int_{-1}^{1}f(x)T_j(x)dx$$

and

$$f(x) = \sqrt{1-x^2}U_2(x) + 4x^5 - 3x^3 = \sqrt{1-x^2}U_2(x) + \frac{1}{4}\left(T_1(x) + 2T_3(x) + T_5(x)\right)$$

By making use of Lemma 1, Lemma 2 and Equation 11, it follows that:

$$\begin{aligned}&A_{00} = \frac{a_0\pi}{2}, A_{10} = 0, A_{20} = 0, A_{30} = 0, B_0 = \frac{a_1}{3}, C_0 = 0, D_0 = -\frac{a_1+a_3}{5}, E_0 = 0,\ \ C_0 = 0, \\ &D_0 = -\frac{a_1+a_3}{5}, E_0 = 0, F_0 = -\frac{a_3}{21}, G_0 = 0, A_{01} = 0, A_{11} = \frac{a_1\pi}{4},\ \ A_{21} = 0, A_{31} = 0, B_1 = 0, \\ &C_1 = \frac{a_0+a_2}{15}, E_1 = -\frac{13a_2}{105}, F_1 = 0, G_1 = -\frac{2}{35}, A_{02} = -\frac{a_0\pi}{4}, A_{12} = 0, A_{22} = \frac{a_2\pi}{4}, \\ &A_{32} = 0, A_{33} = \frac{a_3\pi}{4}, B_3 = 0, C_3 = \frac{a_0+a_2}{7}, D_3 = 0, \quad E_3 - \frac{7a_2}{45}, F_3 = 0,\ \ G_3 = -\frac{94}{315}\end{aligned} \quad (33)$$

equation:

$u(x) = \sqrt{1-x^2}U_2(x) = \sqrt{1-x^2}(4x^2-1)$

Which is identical to the exact solution of Equation 26.

$$\sum_{k=0}^{3} A_{kj} + B_j + C_j + D_j + E_j + F_j = G_j, \quad (32)$$

Example 2

Consider the following singular integral equation:

$$u(x) - \left(⨍_{-1}^{1} \frac{(x+t^2)u(t)}{t-x} dt + \int_{-1}^{1} (x^2+t^3)u(t)\, dt \right) = f(x), \quad (36)$$

where $f(x) = \sqrt{\frac{1-x}{1+x}}(x^2-1) + 2x^4 - 2x^2 - \frac{3}{8}$.

Solution

It can be verified that the solution to Equation 36 is:

$$\sum_{k=0}^{3} a_k \left(w^4(x) W_k(x) - \frac{2}{\pi} ⨍_{-1}^{1} \frac{w^4(t)(x+t^2)W_k(t)}{t-x} dt \right) - \frac{2}{\pi} \sum_{k=0}^{3} a_k \int_{-1}^{1} (x^2+t^3) w^4(t) W_k(t) = f(x)$$

$$u(x) = \sqrt{\frac{1-x}{1+x}}(x^2-1) \quad (37)$$

$$u(x) = w^4(x) \sum_{k=0}^{3} a_k W_k,$$

Let the unknown solution be:

(38)

where $w^4(x) = \sqrt{\frac{1-x}{1+x}}$ is the weight function of the Chebyshev polynomials of the fourth kind, , and are the unknown coefficients to be determined. By substituting the solution (Equation 38) into Equation 36, we get:

(39)

$$t^3 = \frac{1}{8}(W_3(t) - W_2(t) + 3W_1(t) - 3W_0(t))$$

$$t^2 = \frac{1}{4}(W_2(t) - W_1(t) + 2W_0(t))$$

$$t = \frac{1}{2}(W_1(t) - W_0(t))$$

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and the Chebyshev polynomial of the first kind:

$$
\begin{aligned}
t &= T_1(t)\\
t^2 &= \frac{1}{2}\left(T_0(t) + T_2(t)\right)\\
t^3 &= \frac{1}{4}\left(3T_1(t) + T_3(t)\right)\\
t^4 &= \frac{1}{8}\left(3T_0(t) + 4T_2(t) + T_4(t)\right)\\
t^5 &= \frac{1}{16}\left(10T_1(t) + 5T_3(t) + T_5(t)\right)
\end{aligned}
\qquad (41)
$$

$$
\begin{aligned}
&\sum_{k=0}^{3} a_k\left(w^4(x)W_k(x) - \frac{2}{\pi}\left(x \fint_{-1}^{1} \frac{w^4(t)W_k(t)}{t-x}dt + \frac{1}{2}x \fint_{-1}^{1} \frac{\left(w^4(t)\left(T_0(t)+T_2(t)\right)\right)W_k(t)}{t-x}dt\right)\right)\\
&\quad - \frac{2}{\pi}\sum_{k=0}^{3} a_k\left(x^2 \int_{-1}^{1} w^4(t)W_k(t)dt + \frac{1}{8}\int_{-1}^{1} w^4(t)\left(W_3(t) - W_2(t) + 3W_1(t) - 3W_0(t)\right)W_k(t)\,dt\right)\\
&\quad = f(x)
\end{aligned}
\qquad (42)
$$

Make use of the Chebyshev polynomials properties described in Okecha and Onwukwe (2012), the relations (Equations 11 and 15), we obtain:

$$
\begin{aligned}
&\sum_{k=0}^{3} a_k\left(w^4(x)W_k(x) + 2xV_k(x)\right) + a_0\left(V_0(x) + \frac{1}{2}\left(V_2(x) - V_1(x)\right)\right) - 2x^2a_0 +\\
&a_1\left(V_1(x) + \frac{1}{2}\left(V_3(x) - V_0(x)\right)\right) + a_2\left(V_2(x) + \frac{1}{2}\left(V_4(x) + V_0(x)\right)\right) + a_3\left(V_3(x) + \frac{1}{2}\left(V_5(x) + V_1(x)\right)\right)\\
&\quad + \frac{1}{4}\left(3a_0 - 3a_1 + a_2 - a_3\right) = f(x)
\end{aligned}
\qquad (43)
$$

We have following relations of Chebyshev polynomials of the fourth kind: The different integrals in Equation 39 are solvable, when, in the first integral, t2 is expressed as Chebyshev polynomial of the first kind, the second integral, t^3 is expressed in terms of Chebyshev polynomial of the fourth kind. The used of relation (Equation 11) and some useful properties of Chebyshev polynomials defined in Okecha and Onwukwe (2012) will give the exact values of the (40) integrals. Thus, Equation 39 becomes: Simplify the expressions in Equation 43, gives:

$$
\begin{aligned}
&\sum_{k=0}^{3} a_k\left(w^4(x)W_k(x) + 2xV_k(x)\right) + \frac{1}{4}\left(7a_0 - 5a_1 + 3a_2 - a_3\right) + \left(a_1 - \frac{1}{2}\left(a_0 - a_3\right)\right)V_1(x) + \left(a_2 + \frac{1}{2}a_0\right)V_2(x)\\
&\quad + \left(a_3 + \frac{1}{2}a_1\right)V_3(x) + \frac{a_2}{2}V_4(x) + \frac{a_3}{2}V_5(x) - 2x^2a_0 = f(x).
\end{aligned}
\qquad (44)
$$

Multiply each of the terms in Equation 44 by the Chebyshev polynomials of the first kind, $T_j(x), j = 0,1,2,3,$ and integrate from -1 to 1, leads to the equation:

$$
\begin{aligned}
&\sum_{k=0}^{3} a_k\left(\int_{-1}^{1} w^4(x)W_k(x)T_j(x)dx + 2\int_{-1}^{1} xV_k(x)T_j(x)dx\right) + \frac{1}{4}\left(7a_0 - 5a_1 + 3a_2 - a_3\right)\int_{-1}^{1} T_j(x)dx\\
&\quad + \frac{1}{2}\left(2a_1 - a_0 + a_3\right)\int_{-1}^{1} V_1(x)T_j(x)dx + \frac{a_2}{2}\int_{-1}^{1} V_4(x)T_j(x)dx + \frac{1}{2}\left(2a_2 + a_0\right)\int_{-1}^{1} V_2(x)T_j(x)dx\\
&\quad + \frac{1}{2}\left(2a_3 + a_1\right)\int_{-1}^{1} V_3(x)T_j(x)dx + \frac{a_3}{2}\int_{-1}^{1} V_5(x)T_j(x)dx - 2a_0\int_{-1}^{1} x^2T_j(x)dx = \int_{-1}^{1} f(x)T_j(x)dx
\end{aligned}
\qquad (45)
$$

where

$$f(x)=\sqrt{\frac{1-x}{1+x}}(x^2-1)+2x^4-2x^2-\frac{3}{8}=\sqrt{\frac{1-x}{1+x}}(x^2-1)+\frac{1}{8}\left(-5T_0(x)+2T_4(x)\right)$$

Solving the Equation 45 gives a linear system of equations which can be written in a matrix form as follows:

$$\begin{pmatrix} \pi+\frac{7}{2} & -\frac{13}{6} & -\frac{3}{10} & -\frac{119}{30} \\ -\frac{\pi}{2} & \frac{\pi}{2}+\frac{4}{15} & \frac{4}{15} & -\frac{52}{105} \\ -\frac{7}{6} & -\frac{\pi}{2}+\frac{43}{30} & \frac{\pi}{2}+\frac{17}{42} & \frac{31}{42} \\ 0 & \frac{4}{7} & -\frac{\pi}{2}+\frac{4}{7} & \frac{\pi}{2}+\frac{28}{45} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}=\begin{pmatrix} -\frac{\pi}{2}-\frac{77}{60} \\ \frac{\pi}{8} \\ \frac{\pi}{4}+\frac{137}{420} \\ -\frac{\pi}{8} \end{pmatrix} \quad (46)$$

Solving the Equation 46, using simple Matlab command, we obtained:

$$\begin{aligned} a_0 &= -0.500000000000000 \\ a_1 &= -0\,.250000000000000 \\ a_2 &= 1.250000000000000 \\ a_3 &= -0.000000000000000 \end{aligned} \quad (47)$$

Substituting these values of $a_k, k=0,1,2,3$ into Equation 38, the numerical solution for Equation 36 is obtained to be:

$$u(x)=\frac{1}{4}\sqrt{\frac{1-x}{1+x}}(W_2(x)-W_1(x)-2W_0(x))$$

this is identical to the exact solution (Equation 37).

Example 3

Consider the following singular integral equation:

$$xu(x)+\frac{1}{9\pi}\left[\fint_{-1}^{1}\frac{u(t)}{t-x}dt+\int_{-1}^{1}(x^3+xt^2)u(t)dt\right]=f(x)$$

where $f(x)=\frac{x}{\sqrt{1-x^2}}(7+4x^2)+x^3+x$ (48)

Solution

The solution to Equation 48 is:

$$u(x) = \frac{1}{\sqrt{1-x^2}}(7+4x^2) \quad (49)$$

Solving the integral Equation 48 using our developed technique, we set $n = 3$ and the unknown function as:

$$u(x) = \frac{1}{\sqrt{1-x^2}} \sum_{k=0}^{3} a_k T_k(x), \quad (50)$$

Where $T_k(x)$ is the Chebyshev polynomial of the first kind. The used of Chebyshev polynomial of the first kind stem from the fact that its weight function appears in the function $f(x)$ given in Equation 48. By substituting Equation 50 into Equation 48, we get:

$$\sum_{k=0}^{3} a_k \left(\frac{x}{\sqrt{1-x^2}} T_k(x) + \frac{1}{9\pi} \left[\int_{-1}^{1} \frac{T_k(t)}{\sqrt{1-t^2}\,t-x} dt + \int_{-1}^{1} \frac{(x^3+xt^2)T_k(t)}{\sqrt{1-t^2}} dt \right] \right) = f(x) \quad (51)$$

From Equation 7, it follows that:

$$\sum_{k=0}^{3} a_k \left(\frac{x}{\sqrt{1-x^2}} T_k(x) + \frac{1}{9\pi} \left[\int_{-1}^{1} \frac{T_k(t)}{\sqrt{1-t^2}\,t-x} dt + x^3 \int_{-1}^{1} \frac{T_k(t)}{\sqrt{1-t^2}} dt + \frac{x}{2} \int_{-1}^{1} \frac{(T_0(t)+T_2(t))T_k(t)}{\sqrt{1-t^2}} dt \right] \right) = f(x), \quad (52)$$

By applying the properties of Chebyshev polynomials in Equation 51 and simplifying, we obtain: defined in Okecha and Onwukwe (2012) on the integrals

$$\sum_{k=0}^{3} a_k \frac{x}{\sqrt{1-x^2}} T_k(x) + \frac{1}{9} \left[\left(\frac{x}{2} + x^3 \right) a_0 + a_1 + \left(\frac{x}{4} + U_1(x) \right) a_2 + U_2(x) a_3 \right] = f(x), \quad (53)$$

Chebyshev polynomial of the first kind, $T_j(x)$, $j = 0,1,2,3$, expressed the Chebyshev polynomial the first, integrate from -1 to 1, make use of relation (Equation 7), we have the following simplified result:

$$\sum_{k=0}^{3} a_k \int_{-1}^{1} \frac{T_1(x)T_k(x)T_j(x)}{\sqrt{1-x^2}} dx + \frac{a_0}{9} \left[\frac{5}{4} \int_{-1}^{1} T_i(x)T_j(x)\, dx + \frac{1}{4} \int_{-1}^{1} T_3(x)T_j(x)\, dx \right] + \frac{a_1}{9} \int_{-1}^{1} T_j(x) dx + \frac{a_2}{4} \int_{-1}^{1} T_1(x)T_j(x) dx$$
$$+ \frac{a_3}{9} \int_{-1}^{1} \left(T_j(x) + 2T_2(x)T_j(x) \right) dx$$
$$= 10 \int_{-1}^{1} \frac{T_1(x)T_j(x)}{\sqrt{1-x^2}} dx + \int_{-1}^{1} \frac{T_3(x)T_j(x)}{\sqrt{1-x^2}} dx + \frac{7}{4} \int_{-1}^{1} T_1(x)T_j(x) dx + \frac{1}{4} \int_{-1}^{1} T_3(x)T_j(x) dx, \quad (54)$$

In Equation 53, multiply each of the term by the of the second kind in terms of Chebyshev polynomial of From Equation 54, we obtain the system of linear equation which can be written in matrix form as:

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$$\begin{pmatrix} 0 & \frac{\pi}{2}+\frac{2}{9} & 0 & \frac{2}{27} \\ \frac{\pi}{2}+\frac{11}{135} & 0 & \frac{\pi}{4}+\frac{1}{6} & 0 \\ 0 & \frac{\pi}{4}-\frac{2}{3} & 0 & \frac{\pi}{4}-\frac{2}{15} \\ -\frac{1}{35} & 0 & \frac{\pi}{4}-\frac{1}{10} & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5\pi+\frac{16}{15} \\ 0 \\ \frac{\pi}{2}-\frac{16}{35} \end{pmatrix} \quad (55)$$

Solving the system of linear Equation 55, using simple Matlab command, we obtained:

$$\begin{aligned} a_0 &= 9 \\ a_1 &= 0 \\ a_2 &= 2 \\ a_3 &= 0 \end{aligned} \quad (56)$$

Substituting the values of $a_k, k = 0, 1, 2, 3$ into Equation 50, the numerical solution of Equation 25 is obtained to be:

$$u(x) = \frac{1}{\sqrt{1-x^2}}(9T_0(x) + 2T_2(x)) = \frac{1}{\sqrt{1-x^2}}(7 + 4x^2)$$

Which is identical to the exact solution Equation 49.

CONCLUSION

The use of Chebyshev polynomials of the first, second, third and fourth kinds to solve Feedhole integral equation of the second kind with Cauchy singularity has been demonstrated. The unknown function was represented as the sum of the product of unknown constants, the Chebyshev polynomials and their weight function. A linear system of equations was obtained. From Examples 1 and 2, we conclude that our developed method is exact for certain type of Fredholm integral equation of the second kind with Cauchy singularity. Numerical solutions are obtained by MATLAB software.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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